Journal of Statistical Physics, Vol. 66, Nos. 5/6, 1992

Landslides on Sandpiles: Some Moment Relations in One Dimension

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Received July 16, 1991

Several relations between the structure of stable recurrent states and the statistics of avalanches in a one-dimensional sandpile automaton are derived and numerically verified. In particular, it is shown that the average avalanche size is determined by the second rather than the first moment of the distribution of trough distances. The two moments scale differently with system size, which implies multiscaling for the distribution. Moreover, the scaling of edge events (avalanches which fall off the pile) is shown to differ from that of bulk events (avalanches which remain on the pile).

KEY WORDS: Self-organized criticality; sandpiles; multiscaling.

Piles of sand were originally introduced by Bak and co-workers[1] as an illustration of their general theory of self-organization in extended non-equilibrium systems. Dropping sand randomly and slowly onto a table, one eventually ends up in a stationary situation where the sandpile is marginally stable and each subsequent addition of sand can cause landslides on all scales, from a single grain to the size of the pile. Recent theoretical work on the subject ranges from renormalized field theory[2] to developments in the mechanics of powders,[3] partly motivated by experiments[4] on real sandpiles.

The cellular automaton models of Bak *et al.*[1] were systematically studied by Kadanoff *et al.*[5] (KNWZ) and others.[6] The common structural feature of these models, most clearly displayed in the beautiful work of Dhar and co-workers,[7] is the existence of two distinct levels of description: the ensemble of stable recurrent states, and the statistics of

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0022-4715/92/0300-1635$06.50/0

avalanches, which are perturbations of stable states. The two levels are linked by the consistency requirement that avalanches also constitute the dynamics which maps one recurrent state into the next. The ensemble of recurrent states has some similarities with the set of critical configurations in percolation.[7]

In this note I explore the relation between stable recurrent states and avalanches for a one-dimensional sandpile model first introduced by KNWZ[5] and more recently studied by Carlson *et al.*[8] (CCGS). The height of the pile above a finite one-dimensional lattice is given by integer variables h_i, $i=1,..., L$. Sand falls off the open edge at $i=L$, i.e., $h_L=0$ at all times. The system is closed at the left-hand side ($i=1$), so the pile will slope to the right. Sand is added in single grains ($h_i \to h_i+1$) at randomly chosen sites throughout the system. After each addition the stability of the pile is checked. A site becomes unstable if the local slope $z_i=h_i-h_{i+1}$ exceeds a threshold z_c. Then *two* grains of sand fall from site i to site $i+1$, $h_i \to h_i-2$, and $h_{i+1} \to h_{i+1}+2$. Thereby site $i-1$ may become unstable also and the avalanche spreads backward along the pile. The size of the avalanche is determined by the first site i_L to the left of i for which $z_{i_L} \leqslant z_c-2$, a "trough" in the terminology of CCGS. Each of the sites between i_L and i releases two grains of sand, which slide down the pile until the front end of the avalanche encounters another trough at a site $i_R > i$, where the avalanche is stopped. Hence the dynamics which takes one stable recurrent configuration to another can be reduced[8] to the motion, creation, and annihilation of troughs. For events which occur to the right of the last trough on the pile, the avalanche does not stop, but falls off the pile. Such events will be refered to as edge events to distinguish them from bulk events, in which no mass leaves the pile.

CCGS made the remarkable observation that the number of troughs per site $\rho_t=N_t/L$ vanishes as a power law with system size, $\rho_t \sim L^{-\nu}$ in the steady state, with $\nu \simeq 1/3$. This introduces the average distance between troughs $\langle\lambda\rangle=1/\rho_t \sim L^{\nu}$ as a new length scale into the problem, which is large compared to the lattice spacing, but small compared to the system size. Naively, one could expect this length scale to also determine the typical size of avalanches. Here I will argue that the distribution of trough distances $P(\lambda)$ is broad, and that the scale of avalanches is set by

$$\bar{\lambda}=\frac{\langle\lambda^2\rangle}{\langle\lambda\rangle} \sim L^{\tilde{\nu}} \tag{1}$$

where $\tilde{\nu}=1-\nu$ for edge events and $\tilde{\nu}=1/2$ for bulk events. The physical mechanism behind (1) is that *avalanches are more likely to be initiated in regions where the resulting avalanche will be large.* My results show in particular that unless $\nu = 1/2$, which seems very unlikely in view of the numerical results[8] (see below), the distribution of trough distances $P(\lambda)$ displays multiscaling, since different moments of the distribution scale differently with L. This provides a link to the work of KNWZ, who found multiscaling for various statistical distributions of events in one-dimensional sandpiles. Very recently, Manna and Kertész[9] found multiscaling in the distribution of edge events (which is in fact the distribution measured in the experiments of Held *et al.*[4]) of a two-dimensional sandpile model. My arguments only involve elementary statistics and the requirement of stationarity, i.e., the balance between inflow and outflow of mass in the pile, in conjunction with the trough representation developed by CCGS.

We consider a trough-free region Ω of size λ bordered either by two troughs or by one trough and the edge of the system. By definition, sites in Ω are either marginal, $z_i = z_c$, or stable to the addition of one grain, $z_i = z_c - 1$. When a grain is added to a marginal site i there are two possibilities. If site $i-1$ is stable, the avalanche does not spread backward, but remains restricted to a single block of two grains. Since such a move does not change the total number of troughs in the system, CCGS refer to it as a *slide* event.[8] These events constitute a finite fraction[2] $p_s \simeq 1/2$ of all avalanches. It is useful to distinguish the average avalanche mass $\langle m \rangle$, obtained taking *all* events into account, and the average mass of large avalanches $\langle m \rangle_l = (\langle m \rangle - 2p_s)/(1-p_s)$, where slide events are omitted. Large avalanches associated with the *coalescence* of a pair of troughs[8] occur when a grain is dropped onto a pair of marginal sites. They spread to the left border of Ω and slide down to the right border. Thus, for a given value of λ the length of a large avalanche is $\lambda/2$ and its mass is λ on average (recall that an avalanche carries *two* units of mass per site). Let $P(\lambda)$ denote the distribution of λ relative to the ensemble of all stable recurrent configurations. It then follows that the average mass of large avalanches is equal to the average of λ, *conditioned* on the occurrence of an avalanche in the region. This average is not simply equal to the first moment of $P(\lambda)$, since large regions have a better chance to capture a randomly dropped

[2] An important property of this sandpile model is that the parity variables $\sigma_i \equiv z_i \bmod 2$ change only through the addition of grains, not during avalanches. Consequently, the σ_i are uncorrelated in the steady state and take the value 0 or 1 with equal probability. In the absence of troughs, $z_i = z_c$ or $z_c - 1$; hence the slopes are uniquely determined by the parities and are also uncorrelated. This implies that $p = p_s = 1/2$ for large λ. Numerically, I find significant size-dependent deviations which can be described by $p_s \simeq 1/2 + 0.54L^{-0.35}$ for bulk events and $p_s \simeq 1/2 + 0.29L^{-0.24}$ and $p \simeq 0.47$ for edge events. While my conclusions only depend on these probabilities to converge to some nonzero limiting values for $L \to \infty$, which is clearly the case, the size dependence of p_s appears to be a major source of corrections to scaling.

grain. Conditioning on the occurrence of avalanches (i.e., the capture of a grain) defines a new ensemble in which the weight of a region of size λ is multiplied by λ. The normalized probability distribution for the conditioned ensemble is thus

$$\tilde{P}(\lambda) = \frac{\lambda}{\langle \lambda \rangle} P(\lambda) \tag{2}$$

and its first moment is given by (1). We conclude that

$$\langle m \rangle_l = \bar{\lambda} \tag{3}$$

for bulk as well as for edge events. Since the large avalanches dominate the distribution,

$$\langle m \rangle \simeq (1 - p_s) \langle m \rangle_l \simeq \bar{\lambda}/2 \tag{4}$$

asymptotically.

Next I establish the scaling of $\bar{\lambda}$ with L for the edge events. Here λ is the distance of the last trough from the edge. The probability to trigger an avalanche beyond the last trough is $P\langle \lambda \rangle / L$, where $p \simeq 1/2$ is the density of marginal sites in that region (see footnote 2). Since the edge events must transport one grain out of the system for every grain added anywhere on the pile, mass balance requires that the average size of edge avalanches is

$$\langle m \rangle = \frac{L}{P\langle \lambda \rangle} \tag{5}$$

Hence the average mass scales as $L^{1-\nu}$ and using (1) and (4), we expect that $\langle \lambda^2 \rangle \sim L$. I have numerically measured the average mass of edge avalanches and the first and second moments of $P(\lambda)$ for piles of sizes $L =$ 125–4000 (Fig. 1). Power-law fits to the data give $\langle m \rangle = 0.75 L^{0.60}$, $\langle \lambda \rangle =$ $3.3 L^{0.38}$, and $\langle \lambda^2 \rangle = 7.7 L^{0.94}$, in reasonable agreement with the predictions. It should be noted, however, that the high quality of the fit (cf. Fig. 1) rules out a statistical origin of the rather sizeable deviations from the expected behavior. Instead, it seems that the approach to the asymptopic regime is exceedingly slow in these systems, for reasons (see footnote 2) that remain to be understood.

A slightly different mass balance argument is invoked for the bulk events. The number of bulk avalanches triggered by dropping N grains of sand is $O(N)$. Each (large) avalanche covers of the order of $\bar{\lambda}$ sites, hence a typical bulk site is affected by $O(N\bar{\lambda}/L)$ avalanches, each of which carries a mass of $O(\bar{\lambda})$. Mass balance requires the flux through any bulk site to be $O(N)$ and therefore $\bar{\lambda}^2 \sim L$, or

$$\bar{\lambda}_{\text{bulk}} \sim L^{1/2} \tag{6}$$

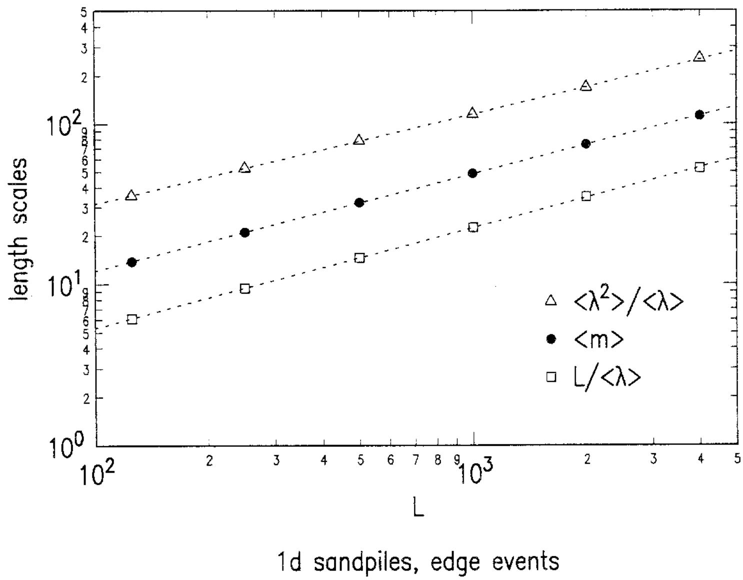

Fig. 1. Length scales associated with edge events. The number of avalanches generated to obtain these data ranges from 2×10^7 for the smallest, to 2×10^5 for the largest piles. The dotted lines are least squares power-law fits to the data. The predicted asymptotic relationship is $\langle \lambda^2 \rangle / \langle \lambda \rangle \simeq 2 \langle m \rangle \simeq 4L / \langle \lambda \rangle$.

It should be emphasized that this argument also applies to sites close to the edge (except for the very last site, $i = L$). This does *not* contradict the previous argument for edge events, since any site $i < L$ has a finite probability to be to the left of the last trough and hence to receive flux from avalanches which do not leave the pile.

Figure 2 shows numerical results for bulk events. Averages were taken over all events and troughs which were located between the first and the last trough on the pile. This procedure smears out the spatial variation of the trough density,[8] but has the advantage of generating better statistics as compared to measuring events occurring at a single site. The results are well described by the power laws $\langle m \rangle = 0.83 L^{0.49}$, $\langle \lambda \rangle = 1.95 L^{0.36}$, and $\langle \lambda^2 \rangle = 3.8 L^{0.84}$, which is considerably closer to the expected behavior than was the case for the edge events. I also note that the average trough distance $\langle \lambda \rangle$ seems to scale with the same exponent $\nu \simeq 0.36$–0.38 for bulk troughs as for the last trough at the edge, which was not necessarily to be expected. It is not clear at present whether the deviation of this effective exponent from the value $\nu \simeq 1/3$ obtained[8] (also J. Krug, unpublished) for the trough density is significant or not.

In summary, I have pointed out the existence of at least three different

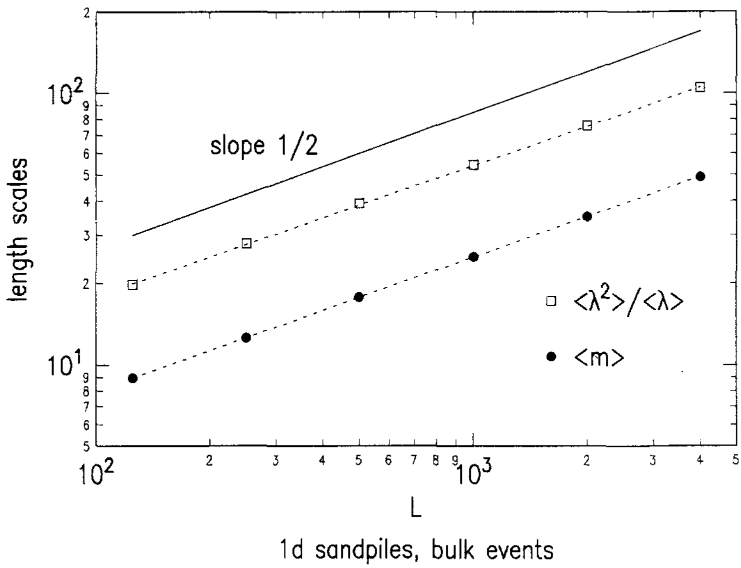

Fig. 2. Length scales associated with bulk events. The number of avalanches generated was between 10^7 for the smallest, and 5×10^6 for the largest piles. The dotted lines are least squares power-law fits and the full line serves to guide the eye. The predicted relationship is $\langle \lambda^2 \rangle / \langle \lambda \rangle \simeq 2 \langle m \rangle \sim L^{1/2}$.

mesoscopic length scales in the one-dimensional sandpile model—the average trough distance, the average size of edge events, and the average size of bulk events—and I have shown how these length scales arise from the large fluctuations in the spatial distribution of troughs. It would be interesting to elucidate further how these results relate to the observed[5] multiscaling in one-dimensional sandpiles, as well as to look for similar complex behavior in higher-dimensional sandpile automata.[6,9]

ACKNOWLEDGMENTS

It is a pleasure to thank Joshua Socolar for many discussions and a critical reading of the manuscript, and Eric Grannan for a useful conversation. I am also grateful to Jennifer Chayes for suggestions which improved the presentation of this paper.

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Communicated by J. Chayes